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Fourier transform and the Verlinde formula for the quantum double of a finite group

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Abstract. We define a Fourier transform S for the quantum double $D(G)$ of a finite group G. Acting on characters of $D(G)$, S and the central ribbon element of $D(G)$ generate a unitary matrix representation of the group $SL(2,\mathbb{Z})$. The characters form a ring over the integers under both the algebra multiplication and its dual, with the latter encoding the fusion rules of $D(G)$. The Fourier transform relates the two ring structures. We use this to give a particularly short proof of the Verlinde formula for the fusion coefficients.

1. Introduction

The quantum double $D(G)$ of a finite group G is a quasi-triangular ribbon Hopf algebra [1] constructed, via Drinfeld's double construction [2], out of the Hopf algebra $C(G)$ of $\mathbb{C}$-valued functions on G. Such quantum doubles arise in physics in orbifold conformal field theories [3] and in the classification of flux-charge composites in the massive phases of (2+1)-dimensional gauge theories [4, 5]. The mathematical structure of $D(G)$ was clarified in [6]. There and in [3] it was also pointed out that the set of irreducible representations of $D(G)$ carries a representation of the group $SL(2,\mathbb{Z})$ by unitary, symmetric matrices. In particular, one has a symmetric and unitary matrix S and a diagonal, unitary matrix T acting on the set of irreducible representations which satisfy the modular relation $(ST)^3 = S^2$ and $S^4 = 1$. Although perhaps surprising from the point of view of Hopf algebras, the appearance of the $SL(2,\mathbb{Z})$ action in the representation theory of $D(G)$ is physically motivated by application of $D(G)$ in orbifold conformal field theories. In particular, it has already been pointed out in [3] that the matrix S plays the role of the Verlinde matrix which diagonalizes the fusion rules in orbifold conformal field theory (for a general review of fusion rules in conformal field theory we refer the reader to [7]). As a result one has a Verlinde formula [8] for integer fusion coefficients in terms of (generally non-integer) matrix elements of the Verlinde matrix S.

A central goal of this paper is to understand the role of the group $SL(2,\mathbb{Z})$ in the representation theory of $D(G)$ without reference to conformal field theory. Our starting point here is the set of characters of irreducible representations of $D(G)$. The group $SL(2,\mathbb{Z})$ acts on this set in a geometrically natural way. We identify two generators S and T of this action (satisfying $(ST)^3 = S^2$ and $S^4 = 1$) which play a natural role in the theory of $D(G)$ and its dual $D(G)^\star$. It was already noted in [6] that the diagonal matrix T is related to the central ribbon element of $D(G)$. As vector spaces, both $D(G)$ and $D(G)^\star$ can be identified with the space $C(G \times G)$ of $\mathbb{C}$-valued functions on $G \times G$, and here we show that S can be extended to an automorphism of the vector space $C(G \times G)$. We prove a convolution theorem for this extension which shows that it has a natural interpretation as a Fourier transform. Returning to the set of characters we show that it can be given a ring structure in two dual ways. One, using the algebra multiplication, is essentially determined by Schur orthogonality relations. The other, using the dual multiplication, encodes the fusion rules of $D(G)$. Our Fourier transform relates the two ring structures, and we use this to give a very short proof of the Verlinde formula for $D(G)$.

Quantum doubles can also be defined for locally compact groups G [9] and we have used a notation which anticipates the generalization of the arguments given here to quantum doubles of locally compact groups. There are a number of technical problems, however, which we intend to address in a future publication. Finally, we observe that the Fourier transform we will define in this paper is related to the non-Abelian Fourier transform defined by Lusztig in the context of finite group theory, see [10, 11], and to the quantum Fourier transform defined by Lyubashenko in [12] and discussed further by Lyubashenko and Majid in [13, 14]. We will clarify the relationship between these definitions and ours in the course of the paper. There are several other places in the literature where Fourier transforms are discussed in the context of quantum groups. The focus of [15, 16] is braided quantum groups and is thus different from ours. In section 3.4 of [17] a Fourier transform is defined for finite-dimensional Hopf algebras. However, when applied to the quantum double of a finite group that definition yields something essentially different from our Fourier transform.

2. The quantum double of a finite group

Let G be a finite group, with invariant measure given by

$$\int_G f(z)\,\mathrm{d}z := |G|^{-1} \sum_{z\in G} f(z). \tag{2.1}$$

We will use delta functions δ_x on G, normalized so that $\delta_x(y) = 0$ if $x \neq y$ and $\delta_x(x) = |G|$.

The quantum double $D(G)$ of a finite group G was first studied in detail in [6]. The definitions we are about to give are equivalent to the ones given there, but we adopt a different notation. The advantage of our notation is that it easily generalizes to the case where G is a locally compact Lie group [9]. As a linear space we identify the quantum double $D(G)$ of G with $C(G \times G)$. On $D(G)$ we have a non-degenerate pairing

$$\langle f_1, f_2\rangle := \int_G\int_G f_1(x, y)\, f_2(x, y)\,\mathrm{d}x\,\mathrm{d}y. \tag{2.2}$$

By this pairing we can also identify the dual $D(G)^\star$ of $D(G)$ with $C(G \times G)$ as a linear space.

On $D(G)$ we have multiplication $\bullet$, identity 1, comultiplication Δ, counit ε, antipode κ

and involution * given by

$$\begin{aligned}
&(f_1 \bullet f_2)(x,y) := \int_G f_1(x,z)\, f_2(z^{-1}xz, z^{-1}y)\,\mathrm{d}z \\
&1(x,y) := \delta_e(y) \\
&(\Delta f)(x_1,y_1;x_2,y_2) := f(x_1x_2,y_1)\delta_{y_1}(y_2) \\
&\varepsilon(f) := \int_G f(e,y)\,\mathrm{d}y \\
&(\kappa f)(x,y) := f(y^{-1}x^{-1}y, y^{-1}) \\
&f^*(x,y) := \overline{f(y^{-1}xy, y^{-1})}.
\end{aligned} \tag{2.3}$$

By duality we have multiplication $\star$, identity ι, comultiplication $\Delta^\star$, counit $\varepsilon^\star$, antipode $\kappa^\star$ and involution $^\circ$ on $D(G)^\star$:

$$\begin{aligned}
&(f_1 \star f_2)(x,y) := \int_G f_1(z,y) f_2(z^{-1}x,y)\,\mathrm{d}z \\
&\iota(x,y) := \delta_e(x) \\
&(\Delta^\star f)(x_1,y_1;x_2,y_2) := f(x_1,y_1y_2)\delta_{x_2}(y_1^{-1}x_1y_1) \\
&\varepsilon^\star(f) := \int_G f(x,e)\,\mathrm{d}x \\
&(\kappa^\star f)(x,y) := f(y^{-1}x^{-1}y, y^{-1}) \\
&f^\circ(x,y) := \overline{f(x^{-1},y)}.
\end{aligned} \tag{2.4}$$

Later, we will also refer to the ribbon algebra structure of $D(G)$. Following the conventions for ribbon Hopf algebras of section 4.2 in [1], we define the universal R-matrix $R \in D(G) \otimes D(G)$:

$$R(x_1,y_1;x_2,y_2) = \delta_e(y_1)\delta_e(x_1y_2^{-1}) \tag{2.5}$$

and the central ribbon element $c \in D(G)$:

$$c(x,y) = \bullet \circ (\kappa \otimes \mathrm{id})(R_{21}) = \delta_e(xy) \tag{2.6}$$

where $R_{21}(x_1,y_1;x_2,y_2) := R(x_2,y_2;x_1,y_1)$. The monodromy element $Q \in D(G) \times D(G)$ is

$$Q(x_1,y_1;x_2,y_2) := (R_{21} \bullet R)(x_1,y_1;x_2,y_2) = \delta_{y_1}(x_2)\delta_{y_2}(x_2^{-1}x_1x_2). \tag{2.7}$$

Together with c, it satisfies the ribbon relation

$$\Delta c = Q^{-1} \bullet (c \otimes c). \tag{2.8}$$

In the representation theory of $D(G)$ and $D(G)^\star$ an important role is played by the Haar functionals $h^\star \colon D(G)^\star \to \mathbb{C}$ and $h \colon D(G) \to \mathbb{C}$, respectively. They are given by

$$h^\star(f) := \int_G f(e,y)\,\mathrm{d}y \qquad \text{and} \qquad h(f) := \int_G f(x,e)\,\mathrm{d}x. \tag{2.9}$$

Here we have chosen the normalization $h^\star(\iota) = h(1) = |G|$. Direct computation shows left and right invariance:

$$((h^\star \otimes \mathrm{id}) \circ \Delta^\star)(f) = h^\star(f)\iota = ((\mathrm{id} \otimes h^\star) \circ \Delta^\star)(f) \tag{2.10}$$

and similarly for h. Furthermore, centrality, positivity and faithfulness of h and $h^\star$ follow from the formulae

$$h(f_1 \bullet f_2^*) = h^\star(f_1 \star f_2^\circ) = \int_G \int_G f_1(x,y)\overline{f_2(x,y)}\,\mathrm{d}x\,\mathrm{d}y \tag{2.11}$$

and

$$h^\star(f \star f^\circ) = \int_G \int_G |f(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y. \tag{2.12}$$

Thus, $C(G \times G)$ has a Hermitian inner product

$$(f_1, f_2) \mapsto h^\star(f_1 \star f_2^\circ) = \langle f_1, \overline{f_2} \rangle. \tag{2.13}$$

From the existence of faithful positive linear functionals on $D(G)$ and $D(G)^\star$ (namely h and $h^\star$) we can conclude that $D(G)$ and $D(G)^\star$ both have a faithful $*$-representation on a finite-dimensional Hilbert space, so they are C^*-algebras. Therefore, the theory of Woronowicz [18] for compact matrix quantum groups holds both for $D(G)$ and $D(G)^\star$. Moreover, this theory simplifies because we are in the finite-dimensional case, see [18], appendix A.2, and [19]. These simplifications are already evident in our explicit results that $\kappa^2 = \mathrm{id}$ and $(\kappa^\star)^2 = \mathrm{id}$, and that h and $h^\star$ are central. Furthermore, in [19] after the proof of proposition 2.2, van Daele gives a formula in terms of dual bases for the Haar functional in the finite-dimensional C^*-algebra case. For $D(G)^\star$ this can be written as

$$h^\star(f) = \text{const.} \sum_{x,y \in G} (f \star \delta_{x,y})(x,y). \tag{2.14}$$

A simple calculation indeed shows that this agrees with (2.9). An analogous formula holds for h.

3. Irreducible representations and their characters

The irreducible representations of the quantum double of a finite group were classified in [6]. We adopt some of the notation used there, but for our description of the representations we follow the approach used in the discussion of the double of a locally compact group in [9, 20]. Thus let $\{C_A\}_{A=0,\ldots,p}$ be the set of conjugacy classes in G, with $C_0 = \{e\}$. In each C_A pick an element g_A and write N_A for the centralizer of g_A in G. For later use it is also convenient to pick, for each $x \in G$, an element $B_x \in G$ such that

$$B_x g_A B_x^{-1} = x \qquad \text{if} \quad x \in C_A. \tag{3.1}$$

Write q_A for the number of irreducible representations of N_A and let $\{\pi_\alpha\}$ be a complete set of such representations. The label α is a positive integer running from 1 (for the trivial representation) to q_A. We denote the carrier spaces by V_α and their dimensions by d_α. Then the irreducible representations π_α^A of $D(G)$ are labelled by pairs (A, α) of conjugacy classes and centralizer representations. The carrier space V_α^A of π_α^A is

$$V_\alpha^A := \{\phi : G \to V_\alpha | \phi(xn) = \pi_\alpha(n^{-1})\phi(x), \forall x \in G, \forall n \in N_A\} \tag{3.2}$$

and the action of an element $f \in D(G)$ is

$$(\pi_\alpha^A(f)\phi)(x) := \int_G f(x g_A x^{-1}, z)\phi(z^{-1}x)\,\mathrm{d}z. \tag{3.3}$$

The set $\{\pi_\alpha^A\}$ is a complete set of mutually inequivalent irreducible matrix $*$-representations of $D(G)$. We write $d_{A,\alpha}$ for the dimension of V_α^A and note that $d_{A,\alpha} = |C_A| \cdot d_\alpha$. Then, after choosing an orthonormal basis of V_α^A, π_α^A can be represented by a matrix $(\pi_\alpha^A)_{ij}$, $i, j = 1, \ldots, d_{A,\alpha}$. The matrix elements $(\pi_\alpha^A)_{ij}$ are elements of $D(G)^\star$ and we write $M_{A,\alpha}$ for the span of the matrix elements $(\pi_\alpha^A)_{ij}$ $(i, j = 1, \ldots, d_{A,\alpha})$. Then it follows from Woronowicz's general theory that $D(G)^\star$ is the orthogonal direct sum of the spaces $M_{A,\alpha}$. Finally, we define the character

$$\chi_\alpha^A = \sum_{i=1}^{d_{A,\alpha}} (\pi_\alpha^A)_{ii}. \tag{3.4}$$

Characters play a fundamental role in the following discussion. From [20] we have the following formula:

$$\chi_\alpha^A(f) = \int_G \int_{N_A} f(zg_Az^{-1}, znz^{-1})\chi_\alpha(n)\,\mathrm{d}n\,\mathrm{d}z. \tag{3.5}$$

Changing integration variables, this can be rewritten as

$$\chi_\alpha^A(f) = \int_G \int_G f(v,w)\mathbf{1}_A(v)\delta_e(vwv^{-1}w^{-1})\chi_\alpha(B_v^{-1}wB_v)\,\mathrm{d}v\,\mathrm{d}w \tag{3.6}$$

where $\mathbf{1}_A$ is the characteristic function of the conjugacy class C_A, normalized so that $\mathbf{1}_A(v) = 1$ if $v \in C_A$ and $\mathbf{1}_A(v) = 0$ otherwise.

By definition, characters are elements of $D^\star(G)$. Using the pairing $\langle , \rangle$ we can therefore identify them with functions on $G \times G$. To do this explicitly we insert a delta function for f,

$$\chi_\alpha^A(x,y) := \chi_\alpha^A(\delta_x \otimes \delta_y) = \delta_e(xyx^{-1}y^{-1})\mathbf{1}_A(x)\chi_\alpha(B_x^{-1}yB_x) \tag{3.7}$$

and reproduce the character formula given in [6]. One checks that the characters enjoy the orthogonality relation

$$\langle \chi_\alpha^A, \overline{\chi_\beta^B} \rangle = |G|\delta_{\alpha\beta}\delta_{AB}. \tag{3.8}$$

As elements of $D(G)^\star$ characters have the property that they are cocentral, i.e. they satisfy $\Delta^\star\chi_\alpha^A = \sigma \circ \Delta^\star\chi_\alpha^A$, where $\sigma : D(G)^\star \times D(G)^\star \to D(G)^\star \times D(G)^\star$ is the flip operation, $\sigma(\lambda,\mu) = (\mu,\lambda)$. Using again the identification of $D(G)^\star$ with $C(G \times G)$, the cocentrality of the characters (3.7) means that their support lies in

$$G_{\mathrm{comm}} := \{(x,y) \in G \times G | xy = yx\} \tag{3.9}$$

and that they are invariant under the simultaneous conjugation of both arguments, in symbols $\chi_\alpha^A(gxg^{-1}, gyg^{-1}) = \chi_\alpha^A(x,y)$ for all $g, x, y \in G$. These properties are also evident in the explicit expression (3.7). We write $C(G_{\mathrm{comm}})$ for the space of functions in $C(G \times G)$ whose support lies in G_{comm}, and $C^0(G_{\mathrm{comm}})$ for the space of functions in $C(G_{\mathrm{comm}})$ which are invariant under the simultaneous conjugation of both arguments. It follows from the remark in [18] after corollary 5.10 that the characters in fact span $C^0(G_{\mathrm{comm}})$. It is instructive to see this explicitly. The dimension of $C^0(G_{\mathrm{comm}})$ is equal to the number of G-conjugacy classes in G_{comm}. To count these, introduce an integer label a for N_A-conjugacy classes in N_A. Since the number of such conjugacy classes is equal to the number of irreducible representations, a runs from 1 to q_A. In the N_A-conjugacy class labelled by a pick an element g_A^a. Then every G-conjugacy class in G_{comm} contains a unique element of the form (g_A, g_A^a) for suitable A and a. The number of conjugacy classes is therefore

$$\dim(C^0(G_{\mathrm{comm}})) = \sum_{A=0}^{p} q_A. \tag{3.10}$$

This is also the number of irreducible representations (A,α) of $D(G)$ and hence the number of characters. By the orthogonality relation, the characters are certainly linearly independent and therefore form an orthogonal basis of $C^0(G_{\mathrm{comm}})$.

The vector space $C^0(G_{\mathrm{comm}})$ is closed under both the multiplication $\bullet$ and the dual multiplication $\star$. This means that the vector space spanned by the characters can be given two algebra structures which are dual to each other. Both these algebras are initially defined over the field $\mathbb{C}$, but in the following section we will show that the structure constants for both algebras are integers. Therefore, the integer linear combinations of characters form a ring over $\mathbb{Z}$ under both the multiplications $\bullet$ and $\star$.

4. Character rings

Again following the general theory given in [18], the matrix elements $(\pi^A_\alpha)_{ij}$ form a complete set of mutually inequivalent irreducible matrix corepresentations π^A_α of $D(G)^\star$. We therefore have

$$\Delta^\star((\pi^A_\alpha)_{ij}) = \sum_k (\pi^A_\alpha)_{ik} \otimes (\pi^A_\alpha)_{kj}. \tag{4.1}$$

The quantum analogues of Schur's orthogonality relations, given in [18], simplify for $D(G)$ because $(\kappa^\star)^2 = \mathrm{id}$. We have

$$\langle(\pi^A_\alpha)_{ij}, (\overline{\pi}^B_\beta)_{kl}\rangle = h^\star((\pi^A_\alpha)_{ij} \star (\pi^B_\beta)^\circ_{kl}) = \delta_{\alpha\beta}\delta_{AB}\delta_{ik}\,\delta_{jl}h^\star(\iota)/d_{A,\alpha}. \tag{4.2}$$

Note that due to a standard theorem in the theory of finite groups, see [21, 22], the ratio

$$n^A_\alpha := h^\star(\iota)/d_{A,\alpha} = |N_A|/d_\alpha \tag{4.3}$$

is an integer. These relations are sufficient to establish the following theorem.

Theorem 4.1. *The map $f \mapsto (n^A_\alpha)^{-1}\chi^A_\alpha \bullet f$ is the orthogonal projection of $D(G)^\star$ onto $M_{A,\alpha}$.*

Proof. We have:

$$\langle\chi^A_\alpha \bullet (\pi^B_\beta)_{ij}, \overline{(\pi^C_\gamma)_{kl}}\rangle = \langle\chi^A_\alpha \otimes (\pi^B_\beta)_{ij}, \Delta^\star\overline{(\pi^C_\gamma)_{kl}}\rangle = \sum_r \langle\chi^A_\alpha, \overline{(\pi^C_\gamma)_{kr}}\rangle\langle(\pi^B_\beta)_{ij}, \overline{(\pi^C_\gamma)_{rl}}\rangle$$

$$= \sum_{m,r}\langle(\pi^A_\alpha)_{mm}, \overline{(\pi^C_\gamma)_{kr}}\rangle\langle(\pi^B_\beta)_{ij}, \overline{(\pi^C_\gamma)_{rl}}\rangle = \delta_{AC}\delta_{BC}\delta_{\alpha\gamma}\delta_{\beta\gamma}\delta_{ik}\delta_{jl}(n^A_\alpha)^2.$$

□

As an immediate consequence we note the following.

Lemma 4.2. *The characters of the quantum double $D(G)$ of a finite group G form a ring over $\mathbb{Z}$ under the multiplication $\bullet$. The multiplication rule is*

$$\chi^A_\alpha \bullet \chi^B_\beta = \delta_{AB}\delta_{\alpha\beta}n^A_\alpha\chi^A_\alpha. \tag{4.4}$$

Next consider the algebra structure of the characters under the dual multiplication $\star$. This is related to the tensor product decomposition into irreducible representations:

$$\pi^A_\alpha \otimes \pi^B_\beta \simeq \bigoplus_{C,\gamma} N^{AB\gamma}_{\alpha\beta C}\pi^C_\gamma. \tag{4.5}$$

We will refer to the non-negative integers $N^{AB\gamma}_{\alpha\beta C}$ as fusion coefficients. By definition of the dual multiplication we have, for $\pi, \rho \in D(G)^\star$ and $f \in D(G)$

$$\langle\pi \otimes \rho, \Delta(f)\rangle = \langle\pi \star \rho, f\rangle. \tag{4.6}$$

Thus, upon taking the trace we find that for all $f \in D(G)$

$$\mathrm{tr}(\pi^A_\alpha \otimes \pi^B_\beta(\Delta(f))) = \sum_{C,\gamma} N^{AB\gamma}_{\alpha\beta C}\,\mathrm{tr}(\pi^C_\gamma(f)). \tag{4.7}$$

Using (4.6) we deduce the following lemma.

Lemma 4.3. *The characters of the quantum double $D(G)$ of a finite group G form a ring over $\mathbb{Z}$ under the multiplication $\star$. The multiplication rule is determined by the fusion coefficients $N^{AB\gamma}_{\alpha\beta C}$:*

$$\chi^A_\alpha \star \chi^B_\beta = \sum_{C,\gamma} N^{AB\gamma}_{\alpha\beta C}\chi^C_\gamma. \tag{4.8}$$

5. $SL(2,\mathbb{Z})$-action, Fourier transform and the Verlinde formula

In this section, a central role is played by a natural action of the group $SL(2,\mathbb{Z})$ of integer unimodular 2×2 matrices on space $C(G_{\text{comm}})$. Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5.1}$$

with a, b, c, d integers such that $ad - bc = 1$, be a generic element of $SL(2,\mathbb{Z})$ and define a right action on G_{comm} via

$$(x, y)M := (x^a y^c, x^b y^d). \tag{5.2}$$

This induces an action of $M \in SL(2,\mathbb{Z})$ on elements $f \in C(G_{\text{comm}})$, which we write as

$$(Mf)(x, y) := f(x^a y^c, x^b y^d). \tag{5.3}$$

The generators

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{5.4}$$

satisfy the modular relation

$$(ST)^3 = S^2 \tag{5.5}$$

and $S^4 = 1$, and both have a natural interpretation within the quantum double $D(G)$. To see this, first note that the actions

$$(Sf)(x, y) = f(y, x^{-1}) \tag{5.6}$$

and

$$(Tf)(x, y) = f(x, xy) \tag{5.7}$$

also make sense for general $f \in C(G\times G)$. We keep the notation (5.6) and (5.7) even when the arguments x and y do not commute. Note that S and T are unitary operators on $C(G\times G)$ with the inner product (2.13). Moreover, one finds that the action of T on $f \in C(G\times G)$ is equal to the multiplication of f by the central element c (2.5):

$$Tf = c \bullet f. \tag{5.8}$$

Acting on $C(G\times G)$, S and T no longer satisfy the modular relation (5.5) but we still have $S^4 = 1$. This last property and the following convolution theorem are our reasons for calling S a Fourier transform.

Theorem 5.1. *If $f, g \in C(G\times G)$ and $\operatorname{supp}(f) \in G_{comm}$ then*

$$S(f \star g) = S(f) \bullet S(g) \qquad \textit{and} \qquad S(f \bullet g) = S(g) \star S(f). \tag{5.9}$$

Proof. If the first factor in a $\bullet$-product has support in G_{comm}, the formulae simplify, yielding

$$\begin{aligned}(S(f) \bullet S(g))(x, y) &= \int_G f(z, x^{-1}) g(z^{-1}y, x^{-1})\, \mathrm{d}z \\ &= (S(f \star g))(x, y).\end{aligned}$$

Similarly we have

$$\begin{aligned}(S(g) \star S(f))(x, y) &= \int_G g(y, z^{-1}) f(y, x^{-1}z)\, \mathrm{d}z \\ &= \int_G f(y, w)\, g(y, w^{-1}x^{-1})\, \mathrm{d}w \\ &= (S(f \bullet g))(x, y)\end{aligned}$$

where we have again used $\mathrm{supp}(f) \in G_{\mathrm{comm}}$ and exploited the invariance of the measure under the change of integration variable $w = x^{-1}z$. $\square$

Since S leaves the space of functions with support in G_{comm} invariant, we deduce the following

Corollary 5.2. *If* $f, g \in C(G \times G)$ *and* $\mathrm{supp}(f) \in G_{comm}$ *then*

$$S^{-1}(f \bullet g) = S^{-1}(f) \star S^{-1}(g) \qquad \textit{and} \qquad S^{-1}(g \star f) = S^{-1}(f) \bullet S^{-1}(g). \tag{5.10}$$

At this point it is instructive to make contact with related discussions of Fourier transforms in the literature. By defining a slight variant of the operator S we can establish a connection with the non-Abelian Fourier transform given by Lusztig in [10, 11]. For $f \in C(G \times G)$ put

$$(Uf)(x, y) := f(y, x) \qquad (J_1 f)(x, y) := f(x^{-1}, y) \qquad (J_2 f)(x, y) := f(x, y^{-1}). \tag{5.11}$$

Then $U = J_2 S = S J_1$. The operators U, J_1, J_2 correspond to 2×2 matrices with integer entries but with determinant -1. Note also that the operators U, J_1 and J_2, like S, leave the space of functions with support in G_{comm} invariant and commute with conjugations by elements of G. We can therefore in particular consider the restriction of U to $C^0(G_{\mathrm{comm}})$. It turns out that the matrix elements of U with respect to the basis of characters formally coincide with Lusztig's Fourier kernel:

$$\begin{aligned} U^{BA}_{\beta\alpha} :&= |G|^{-1} \langle U \chi^A_\alpha, \overline{\chi^B_\beta} \rangle \\ &= \frac{1}{|N_A|\,|N_B|} \sum_{\substack{g \in G \\ g_A g g_B g^{-1} = g g_B g^{-1} g_A}} \chi_\alpha(g g_B g^{-1}) \overline{\chi_\beta(g^{-1} g_A g)} \\ &=: \{(g_A, \alpha), (g_B, \beta)\}. \end{aligned} \tag{5.12}$$

However, Lusztig takes $\{\,,\}$ with values in the field $\overline{\mathbb{Q}}_l$, i.e., in an algebraic completion of the field $\mathbb{Q}_l$ of l-adic numbers. He also has a bar operation on $\overline{\mathbb{Q}}_l$.

Straightforward computations shows that the following analogues of theorem 5.1 hold. For $f, g \in C(G \times G)$ we have

$$\begin{aligned} &J_1(f) \star J_1(g) = J_1(g \star f) \qquad J_1(f) \bullet J_1(g) = J_1(f \bullet g) \\ &\qquad \text{and} \qquad J_2(f) \star J_2(g) = J_2(f \star g). \end{aligned} \tag{5.13}$$

If $\mathrm{supp}(f) \in G_{\mathrm{comm}}$ one checks that furthermore

$$U(f) \star U(g) = U(f \bullet g) \qquad \text{and} \qquad U(f) \bullet U(g) = U(f \star g). \tag{5.14}$$

Finally, if $\mathrm{supp}(f)$ and $\mathrm{supp}(g) \in G_{\mathrm{comm}}$ then

$$J_2(f) \bullet J_2(g) = J_2(g \bullet f). \tag{5.15}$$

In the abstract setting of tensor categories, a quantum Fourier transform was defined by Lyubashenko in [12] and discussed further in [13, 14] for finite-dimensional factorizable ribbon Hopf algebras. Applied to $D(G)$ and in our notation their formula for the Fourier transform $\tilde{S}$ of an element f of $D(G)$ reads

$$\tilde{S} f := (1 \otimes h) \circ (R^{-1} \bullet (1 \otimes f) \bullet R_{21}^{-1}). \tag{5.16}$$

An explicit calculation shows that

$$(\tilde{S} f)(x, y) = f(x y^{-1} x^{-1}, x). \tag{5.17}$$

and, therefore, the relation between $\tilde{S}$ and S can be expressed via

$$\tilde{S} = \kappa \circ S. \tag{5.18}$$

The fourth power of the map $\tilde{S}$ is not equal to the identity, but if, following [13], one defines $\tilde{T} = T^{-1}$ one finds that the modular relation $(\tilde{S}\tilde{T})^3 = \tilde{S}^2$ is satisfied. Convolution formulae similar to ours can be proven for the map (5.16). Again, at least one of the two elements f and g to be multiplied has to have support in G_{comm}. Finally, we observe that restricted to $C(G_{\mathrm{comm}})$, the map $\tilde{S}$ agrees with our S^{-1}.

For the rest of this paper we focus our attention on the space $C^0(G_{\mathrm{comm}})$. In particular, we consider the restriction of the map S to $C^0(G_{\mathrm{comm}})$, and again denote it by S. The characters form a natural orthogonal basis of the space $C^0(G_{\mathrm{comm}})$, and we define the matrix $S^{BA}_{\beta\alpha}$ as the matrix representing the map S on the basis of characters:

$$S^{BA}_{\beta\alpha} := |G|^{-1}\langle S\chi^A_\alpha, \overline{\chi^B_\beta}\rangle. \tag{5.19}$$

Here the normalization is chosen so that $S\chi^A_\alpha = \sum_{B,\beta} S^{BA}_{\beta\alpha}\chi^B_\beta$. The matrix $S^{BA}_{\beta\alpha}$ is unitary because the map S is. Using the explicit expression for the characters (3.7) one finds the following formula, first given in [6]:

$$S^{BA}_{\beta\alpha} = \int_G\int_G \delta_e(xyx^{-1}y^{-1})\mathbf{1}_A(x)\mathbf{1}_B(y)\overline{\chi}_\alpha(B_x^{-1}yB_x)\overline{\chi}_\beta(B_y^{-1}xB_y)\,\mathrm{d}x\,\mathrm{d}y. \tag{5.20}$$

This expression shows that the matrix $S^{AB}_{\alpha\beta}$ is symmetric, $S^{AB}_{\alpha\beta} = S^{BA}_{\beta\alpha}$. Since it is also unitary its inverse is given by its complex conjugate. We can also read off the useful relation

$$S^{0A}_{1\alpha} = \frac{1}{n^A_\alpha}. \tag{5.21}$$

Armed with this notation, we can now use the convolution theorem to relate the $\bullet$ and $\star$-ring structures of the characters. The result is the Verlinde formula.

Theorem 5.3 (Verlinde formula). *Acting on characters, the inverse Fourier transform S^{-1} diagonalizes the fusion rules of $D(G)$. The fusion coefficients can be expressed in terms of the matrix $S^{AB}_{\alpha\beta}$:*

$$N^{AB\gamma}_{\alpha\beta C} = \sum_{D,\delta}\frac{S^{DA}_{\delta\alpha}S^{DB}_{\delta\beta}\overline{S}^{CD}_{\gamma\delta}}{S^{0D}_{1\delta}}. \tag{5.22}$$

Proof. It follows from the definition of S and from lemma 4.2 that

$$S\chi^A_\alpha \bullet \chi^B_\beta = \frac{S^{BA}_{\beta\alpha}}{S^{0B}_{1\beta}}\chi^B_\beta.$$

Now apply S^{-1} to both sides and use the first formula in corollary 5.2 to obtain

$$\chi^A_\alpha \star S^{-1}\chi^B_\beta = \frac{S^{AB}_{\alpha\beta}}{S^{0B}_{1\beta}}S^{-1}\chi^B_\beta$$

yielding the diagonalized fusion rules, with eigenvalues $S^{AB}_{\alpha\beta}/S^{0B}_{1\beta}$. A quick derivation of the formula for the fusion coefficients follows again from the definition of S and from lemma 4.2:

$$S\chi^A_\alpha \bullet S\chi^B_\beta = \sum_{D,\delta}\frac{S^{DA}_{\delta\alpha}S^{DB}_{\delta\beta}}{S^{0D}_{1\delta}}\chi^D_\delta.$$

Again apply S^{-1} to both sides and use the first formula in corollary 5.2 to obtain

$$\chi^A_\alpha \star \chi^B_\beta = \sum_{C\gamma}\left(\sum_{D,\delta}\frac{S^{DA}_{\delta\alpha}S^{DB}_{\delta\beta}\overline{S}^{CD}_{\gamma\delta}}{S^{0D}_{1\delta}}\right)\chi^C_\gamma.$$

Comparing this expression with (4.8) shows that the expression in brackets is equal to the fusion coefficient $N^{AB\gamma}_{\alpha\beta C}$. □

Remark. There is an interesting connection with Lusztig's matrix $U^{AB}_{\alpha\beta}$ (5.12) here. We find that $\overline{U^{AB}_{\alpha\beta}} = U^{BA}_{\beta\alpha} = (U^{-1})^{BA}_{\beta\alpha}$ and that $U^{AB}_{\alpha\beta} = S^{AB}_{\alpha\overline{\beta}}$. Verlinde's formula can also be expressed in terms of U:

$$\chi^A_\alpha \star U\chi^B_\beta = \frac{U^{BA}_{\beta\alpha}}{U^{0B}_{1\beta}} U\chi^B_\beta \tag{5.23}$$

and

$$N^{AB\gamma}_{\alpha\beta C} = \sum_{D,\delta} \frac{U^{DA}_{\delta\alpha} U^{DB}_{\delta\beta} U^{CD}_{\gamma\delta}}{U^{0D}_{1\delta}}. \tag{5.24}$$

The simplicity of our proof of the Verlinde formula shows that the Fourier transform S is a very natural tool for proving the Verlinde formula for $D(G)$. While we have restricted attention to a particular ribbon Hopf algebra here, we have tried to indicate as far as possible how our definitions and equations for $D(G)$ can be formulated using only natural operations (such the Haar measure, the antipode, the universal R-matrix or the central ribbon element) which exist for a large class of (quasi-triangular ribbon) Hopf algebras. More generally it is natural to ask for which class of (quasi) Hopf algebras a Fourier transform with analogous properties can be defined. In view of the tight connection between fusion rules in rational conformal field theory and tensor decomposition rules in (quasi-)Hopf algebras (see e.g. [23], or [7] for a review) such a generalized Fourier transform, if it exists, could be expected to play an important role in both Hopf algebra theory and conformal field theory.

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References

[1] Chari V and Pressley A 1994 *Quantum Groups* (Cambridge: Cambridge University Press)
[2] Drinfel'd V G 1986 Quantum groups *Proc. Int. Congress Mathematicians (Berkeley, 1986)* (Providence, RI: American Mathematical Society) pp 798–820
[3] Dijkgraaf R, Vafa C, Verlinde E and Verlinde H 1989 The operator algebra of orbifold models *Commun. Math. Phys.* **123** 485–526
[4] Bais F A, van Driel P and de Wild Propitius M 1992 Quantum symmetries in discrete gauge theories *Phys. Lett.* B **63** 63
[5] de Wild Propitius M and Bais F A 1999 Discrete gauge theories *Particles and Fields* ed G Semenoff and L Vinet (Berlin: Springer) pp 353–431
[6] Dijkgraaf R, Pasquier V and Roche P 1990 Quasi Hopf algebras, group cohomology and orbifold models *Nucl. Phys.* B (Proc. Suppl.) **18** 60–72
[7] Fuchs J 1994 Fusion rules in conformal field theory *Fortsch. Phys.* **42** 1–48
[8] Verlinde E 1988 Fusion rules and modular transformations in 2D conformal field theory *Nucl. Phys.* B **300** 360–76
[9] Koornwinder T H and Muller N M 1997 The quantum double of a (locally) compact group *J. Lie Theory* **7** 33–52
Koornwinder T H and Muller N M 1998 *J. Lie Theory* **8** 187 (erratum)
[10] Lusztig G 1979 Unipotent representations of a finite chevalley group of type E_8 *Q. J. Math. Oxford* **30** 315—38

[11] Lusztig G 1984 *Characters of Reductive Groups over a Finite Field* (Princeton, NJ: Princeton University Press)
[12] Lyubashenko V 1995 Modular transformations and tensor categories *J. Pure Appl. Algebra* **98** 279–327
[13] Lyubashenko V and Majid S 1994 Braided groups and quantum Fourier transform *J. Algebra* **166** 506–28
[14] Lyubashenko V and Majid S 1992 Fourier transform identities in quantum mechanics and the quantum line *Phys. Lett.* B **284** 66–70
[15] Kempf A and Majid S 1994 Algebraic q-integration and Fourier theory on quantum braided spaces *J. Math. Phys.* **35** 6802–37
[16] Bespalov Y, Kerler T, Lyubashenko V and Turaev V 1997 Integrals for braided Hopf algebras *Preprint* q-alg/9709020 or math.QA/9709160
[17] Chryssomalakos C 1997 Remarks on quantum integration *Commun. Math. Phys.* **184** 1–25
[18] Woronowicz S L 1987 Compact matrix pseudogroups *Commun. Math. Phys.* **111** 613–65
[19] van Daele A 1997 The Haar measure on finite quantum groups *Proc. Am. Math. Soc.* **125** 3489–500
[20] Koornwinder T H, Bais F A and Muller N M 1998 Tensor product representations of the quantum double of a compact group *Commun. Math. Phys.* **198** 157–86
[21] Serre J P 1967 *Représentations Linéaires des Groupes Finis* (Paris: Hermann)
[22] James G and Liebeck M 1993 *Representations and Characters of Groups* (Cambridge: Cambridge University Press)
[23] Fuchs J, Ganchev A and Vecsernyes P 1995 Rational Hopf algebras: polynomial equations, gauge fixing, and low dimensional examples *Int. J. Mod. Phys.* A **10** 3431–76